Random Vibrations of Bladed-Disk Assemblies Under Cyclostationary Excitation

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Random vibration of bladed-disk assemblies is studied. The blade excitation is modeled as a nonstationary random process whose statistics vary periodically in time. To the best of our knowledge, this is the first stochastic model that accounts for the unsteadiness of the turbulent forces applied to the blades. It is shown that a conventional approach, which treats the blade excitation as a stationary process, is likely to underestimate the response of the blades. Moreover, the amplitude of the nonstationary random part of the response is found to be comparable to the deterministic part. Consequently, it is important to model the flow around the blades as a nonstationary stochastic process.

Nomenclature

$\bar{F}$ = mean lift force
F_k = force to kth blade
F_L = lift force
$H_i(\omega)$ = ith element of transfer matrix/vector of the bladed-disk assembly
$R_{F_\ell F_k}(t + \tau, t)$ = cross-correlation function of forces F_ℓ and F_k
$R_r(t + \tau, t)$ = autocorrelation function of the response
r = mean radius of bladed-disk assembly
S = number of stators
V_i = parallel component of instantaneous inflow velocity
$\bar{V}_i$ = mean inflow velocity
V_i^P = perpendicular component of instantaneous inflow velocity
V_{rel1} = instantaneous relative velocity
V_X = velocity parallel to an airfoil
V_Y = velocity perpendicular to an airfoil
α = angle of attack
$\alpha(t)$ = modulating amplitude in the expression for the random part of the parallel inflow velocity
α_1 = mean angle of attack
β = zero lift angle
β_1 = mean angle of relative velocity
$\gamma(t)$ = modulating amplitude in the expression for the random part of the perpendicular inflow velocity
ΔF_i = derivative of the blade force with respect to the inflow mean velocity for $V_i = \bar{V}_i$ and $V_i^P = 0$
ΔF_i^P = derivative of the blade force with respect to the perpendicular velocity for $V_i = \bar{V}_i$ and $V_i^P = 0$
δ = rotor blade angle
θ_1 = stator blade angle
ω_a = angular velocity of rotation of the rotor
ω_0 = fundamental carrier frequency for the random part of the inflow velocity

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Introduction

THE design of bladed-disk assemblies, which are the primary structural components in turbines, propellers, and compressors, is a challenging task. The blades of such components rotate in nonuniform flow fields. Therefore, they are subjected to unsteady aerodynamic or hydrodynamic loads. One of the problems which is faced by designers of such components is the prediction of their response to these loads.

The majority of studies on forced response of such structures assume that the aerodynamic or hydrodynamic forces are deterministic. This is equivalent to considering only the ensembled mean of the blade excitation. In reality, the excitation varies randomly in time and also from blade to blade. This is primarily due to the random nature of the turbulent flow around the blades, but it can also be due to imperfections in the geometry of the blades and their nonuniform deterioration.

The ensemble mean and the root mean square (rms) of the instantaneous flow velocity at a fixed point behind a propeller, which operates in a wind tunnel, are depicted in Fig. 1. (This figure is reproduced from Ref. 1.) The same quantities for a fixed point behind the stator of a turbine, are plotted in Fig. 2, which is extracted from Ref. 2.

Some conclusions on the general characteristics of the flow, which is encountered by the blades, can be drawn from Fig. 2. First, the characteristics of the flow change periodically in time due to the stator. Second, the intensity of the random fluctuation in the flow, which is measured by the rms of instantaneous flow velocity, also changes periodically in time. Consider a disk positioned behind the stator, but in front of the rotor. The intensity of turbulence is higher at a point exactly behind a stator blade than it is at a point between two stator blades. Therefore, the velocity fluctuates more vigorously in the wakes behind the stator blades than it does between them. Since the intensity of random fluctuation varies in time, statistical quantities which are used to describe this fluctuation, such as the instantaneous velocity rms, also vary periodically in time. Finally, the blade forces vary in the same manner because they are functions of the velocity.

The blade excitation is a random process which is nonstationary because the statistical character of the flow, and the blade excitation vary in time. Furthermore, since the variation is periodic in time, the statistics of the blade forces are also periodic. These random processes are called cyclostationary, and they have been studied by Gudzenko,[3] Gladyshev,[4] Franks,[5] and Ogura.[6]

The statistical properties and the response of linear systems driven by cyclostationary excitation have been studied in statistical communication theory.[7] Nikolaidis studied the vibration of marine diesel engine propulsion shafting systems under

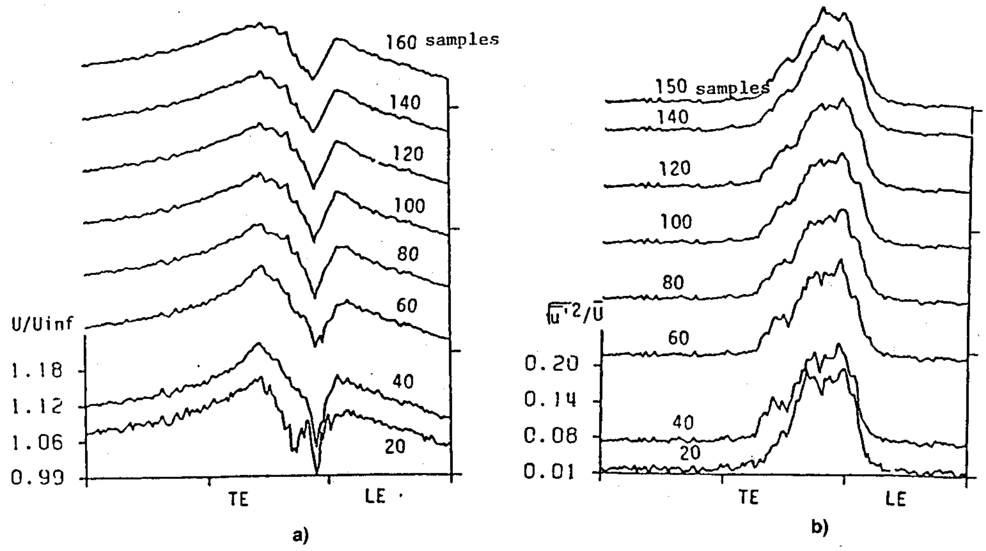

Fig. 1 Instantaneous flow velocity at a fixed point behind a propeller: a) ensembled mean and b) ensembled root mean square. Notes: TE and LE denote the trailing and the leading edge, respectively. The means have been estimated by averaging a number of sample values.

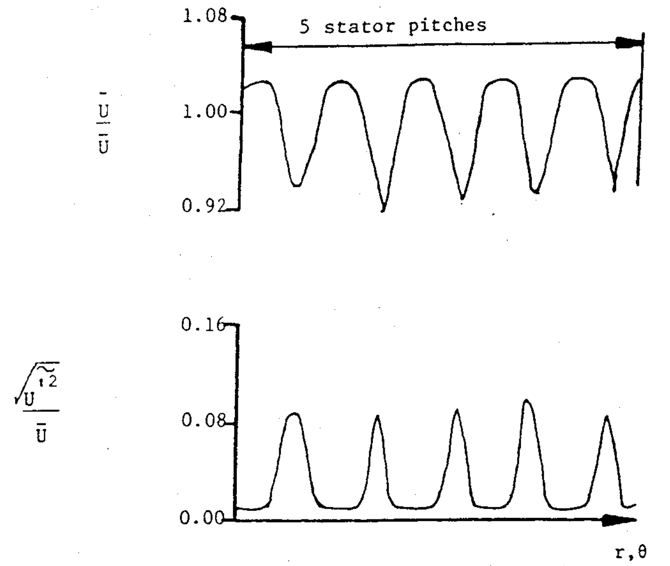

Fig. 2 Ensembled mean and ensembled root mean square of instantaneous flow velocity.

the random excitation from the cylinders and the propeller, which is also a cyclostationary process.[8-11]

The response of a bladed disk assembly to random turbulent excitation is important because of the following reasons:

1) The intensity of fluctuation of the flow velocity due to turbulence is large. For example, the maximum rms of the velocity was found roughly 9% of the corresponding mean value, in Ref. 2, Fig. 2. Kotb's results also showed that the rms might reach 20% for the case of a propeller operating in a wind tunnel.[1]

2) The intensity of turbulence is even higher if a turbine operates at off design conditions. This is particularly true for the case that the blades stall.

3) Although a large portion of the energy of turbulent excitation is in high frequencies, there are also large eddies whose size is comparable to the blades. The loads and the response due to such large eddies might be important compared to their deterministic counterparts. The response of bladed disk assemblies to turbulent excitation has been investigated in Refs. 12 and 13.

The objective of this paper is to demonstrate that it is important to account for the random and the nonstationary character of the blade forces. For this purpose, a stochastic model is developed for the blade excitation. The blade force is modeled by a cyclostationary random process, and the statistical correlation between blade forces is taken into account. The response of the blades, which in our case is their displacement, is calculated. We compare the following responses:

a) The response to deterministic aerodynamic excitation.

b) The response to random excitation, which is calculated when the blade excitation is modeled as cyclostationary.

c) The response to random excitation, which is calculated by an alternative simplified approach, which treats the excitation as a stationary process.

The results from these comparisons allowed us to assess the importance of the random turbulent part of the excitation and also its unsteadiness.

Background

The last decade has been active for research in the area of vibrations of propeller, windmill, and turbine blades that reflects an increased awareness of the complexity and importance of the topic. Most of the research in this area was based on the assumption that parameters such as geometrical properties of the bladed-disk assembly, damping, and blade excitation can be exactly predicted. Consequently, the blade forces were modeled as deterministic functions which are periodic in time. Furthermore, all blade forces have the same amplitude.

Rao,[14] and Ewins[15] gave detailed surveys on the forced vibration response of bladed-disk assemblies. Most studies treating the forced vibration problem, such as in Ref. 16, assume that the blade forces are deterministic. They start from rather simple models for the blades. For example, Griffin and Hoosac[17] used a three-degree-of-freedom system for each blade. The mass and stiffness in this model were equal to the modal mass and stiffness of the blade at the natural frequency of interest. More refined blade models have been employed in Ref.[14]

In some studies, the randomness in the excitation has been accounted for,[12] but the models employed were stationary. Moreover, the correlation between the forces applied to each blade was not taken into account. Booth and Fleeter[13] studied the forced vibration response of turbine blades to turbulence induced random excitation. The blade forces were modeled as white noise. Each blade was represented by a linear system which has two degrees of freedom corresponding to the translational and rotational modes. It was demonstrated that the response increases as the flutter margin decreases. However, the dynamic model used in this study was incomplete because the coupling between blades due to the disk and the shrouds, as well as the statistical correlation between the blade forces, were neglected. Finally, the blade forces were modeled as stationary random processes, which means that the effect of the temporal variation of the statistical properties of the forces was also neglected.

Blade Excitation

Introduction

In this section, a statistical model of the blade forces is developed.

First, the turbulent flow field after the stator is considered with respect to a fixed coordinate system. This random field is described through its bivariate autocorrelation function. The two independent variables are the angle, which defines the point under consideration and it is measured from some reference position and the time. An autocorrelation function of exponential type is assumed for both the temporal and spatial variations of the inflow velocity. The next step is to derive the statistics of the relative velocity field with respect to a relative coordinate system attached to a blade. Finally, the statistical character of the blade forces is described through their autocorrelation and cross-correlations, which are derived from the statistics of the inflow field after the stator. The airfoil of the blade is modeled as a flat-plate, and the lift coefficient is assumed to be proportional to the angle of attack. Furthermore, the blade force is linearized with respect to the flow velocity about its mean value in order to further simplify the approach. Although these assumptions may be oversimplifying, they are acceptable here because our goal is merely to demonstrate the importance of the unsteadiness of turbulent excitation. Moreover, the assumption that the lift is proportional to the angle of attack should provide a reasonable first approximation of the blade forces. The reason is that blade forces are primarily due to large eddies, which correspond to low frequencies. The change in the angle of attack due to these eddies is not rapid.

Statistical Model of the Flow Field Which Is Encountered by the Rotor Blades

In a turbulent flow, the instantaneous flow velocity changes randomly in both time and space. Therefore, in this particular problem, the inflow velocity V_i is a four-dimensional random field, the first three dimensions corresponding to the three spatial coordinates and the fourth to the time. Consider a cylindrical coordinate system, such as that in Fig. 3, which is fixed in space with its origin located at the center of the disk. The following assumptions are made regarding the velocity field $V_i(r, \theta, Z, t)$:

1) The velocity does not change significantly in the radial direction.

2) The variation of the velocity in the Z direction is not important.

As it can be observed from Fig. 4, the instantaneous velocity at any point in front of the rotor disk can be decomposed into two perpendicular components V_i, and V_i^P. V_i is parallel to the freestream velocity. If turbulence were neglected, V_i would be a deterministic function changing periodically in θ with period equal to 2π divided by the number of stators, while V_i^P would be zero. Due to the turbulent nature of the flow, both components of the instantaneous velocity consist of two parts, a deterministic and a random one. As mentioned earlier, the intensity of the random fluctuation of the velocity is not constant with θ, but it varies in a periodic manner. One cycle of such variation corresponds to an angle equal to 2π divided by the number of stators. Since the velocity does not fluctuate uniformly over a disk after the stator, a homogeneous random field in θ is not suitable for describing its randomness.

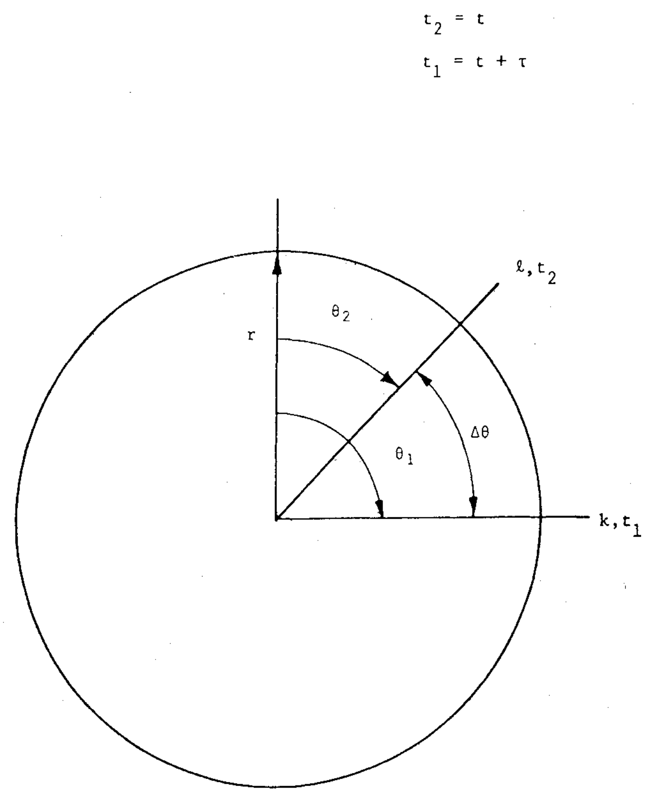

Fig. 3 Fixed coordinate system.

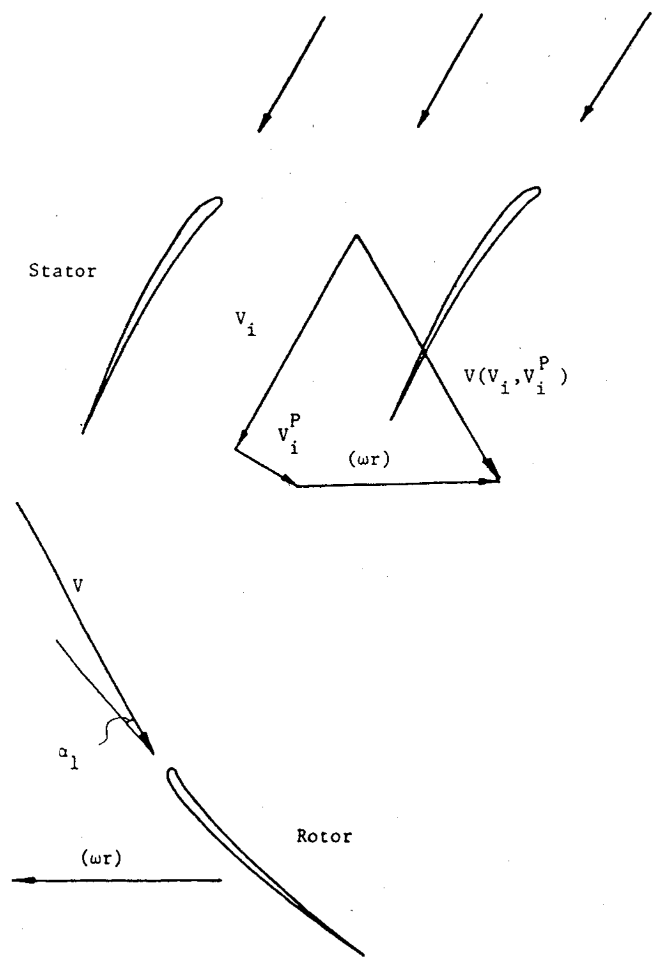

Fig. 4 Velocity triangles.

The following equations are suggested for the two components of the velocity:

$$V_i(t, \theta) = P_1(\theta) + \alpha'(t, \theta)P_2(\theta)$$
$$V_i^p(t, \theta) = \gamma'(t, \theta)P_2(\theta). \quad (1)$$

$P_1(\theta)$ and $P_2(\theta)$ are periodic functions in θ, and $\alpha'(t, \theta)$ and $\gamma'(t, \theta)$ are random fields with zero mean. Here, the random parts of the two velocity components, which are due to turbulence, are modeled as amplitude modulated functions of θ. The periodic function $P_2(\theta)$ is used to expand or constrain the fluctuation of the velocity which is stronger behind the stator blades than it is between them.

For a particular turbine, the carriers $P_1(\theta)$ can be estimated by measuring the ensemble mean and rms of the instantaneous flow velocity at various positions after the stator. Fig. 2 is an example of results from such measurements. Here $P_1(\theta)$ is the ensembled mean of the instantaneous flow velocity, which is also called phase locked mean, and is denoted by $\bar{U}$. $P_2(\theta)$ is the ratio of the instantaneous rms of the velocity and its average value over the circumference.

Random fields $\alpha'(t, \theta)$ and $\gamma'(t, \theta)$ are assumed to be zero mean, Gaussian, and as such they are completely determined by their autocorrelation functions. In this study, an autocorrelation function of exponential form is assumed for both time and space variations. Thus, the autocorrelation function of $\alpha'(t, \theta)$ and $\gamma'(t, \theta)$ can be written as follows:

$$R_{\alpha'}(\tau, \Delta\theta) = \sigma_\alpha^2 e^{-C_1|\tau|} e^{-C_2|\Delta\theta|r}$$
$$R_{\gamma'}(\tau, \Delta\theta) = \sigma_\gamma^2 e^{-C_3|\tau|} e^{-C_4|\Delta\theta|r} \quad (2)$$

where σ_α and σ_γ are inversely proportional to the scale of turbulence in time and space. For example, the inverse of C_1 is equal to one decorrelation period of the parallel to the freestream velocity component, while the inverse of C_2 is equal to one decorrelation distance measured in radians.

Next, consider the k^{th} blade rotating with angular frequency ω_a. The two components of the velocity encountered by the k^{th} blade at time t are

$$V_{i_k}(t) = P_{1_k}(t) + \alpha_k(t)P_{2_k}(t) \quad \text{and} \quad V_{i_k}^p(t) = \gamma_k(t)P_{2_k}(t) \quad (3)$$

where

$$P_{1_k}(t) = P_1(\omega_a t - \theta_k)$$
$$P_{2_k}(t) = P_2(\omega_a t - \theta_k)$$
$$\alpha_k(t) = \alpha'(t, \omega_a t - \theta_k)$$
$$\gamma_k(t) = \gamma'(t, \omega_a t - \theta_k)$$
$$\theta_k = \frac{2\pi}{N}(k - 1) \quad (4)$$

Considering blades k and ℓ, the cross-correlation of $\alpha_k(t)$ and $\alpha_\ell(t)$ is

$$R_{\alpha_\ell \alpha_k}(\tau) = \sigma_\alpha^2 e^{-C_1|\tau|} e^{-C_2\omega_a r\left|\tau - \frac{2\pi(k-\ell)}{\omega_a N}\right|} \quad (5)$$

The autocorrelation of $\alpha_k(t)$ is obtained by letting $k = \ell$ in Eq. (5). Similarly, the equation for the cross-correlation of $\gamma_k(t)$ and $\gamma_\ell(t)$, $R_{\gamma_k\gamma_\ell}(\tau)$, is obtained by substituting C_1 and C_2 by C_3 and C_4, respectively, in Eq. (5). Finally, it is assumed that no correlation exists between the parallel and the perpendicular velocity components, i.e.,

$$R_{\alpha_i\gamma_j}(\tau) = 0 \quad \text{for} \quad i, j = 1, \ldots, N$$

Statistical Model of Blade Forces

The lift on an airfoil with chord length ℓ can be evaluated using the following equation:

$$F_L = \frac{1}{2} C_L \rho \ell V_\infty^2 \quad (6)$$

where the lift coefficient, C_L, can be approximated for small angles of attack, α, by

$$C_L = 2\pi(\alpha + \beta) \quad (7)$$

where β is the zero lift angle.

The lift force to the blade in Fig. 5 is

$$F_K \approx \rho\pi\ell[\beta V_X^2 + V_X V_Y] \quad (8)$$

where

$$V_X = V_{rel_1}\cos(\alpha + \alpha_1) \quad \text{and} \quad V_Y = V_{rel_1}\sin(\alpha + \alpha_1), \quad (9)$$

and V_{rel_1} is the instantaneous relative velocity as seen by the rotor blade.
th the aid of Figure 4 it can be shown that

$$V_{rel_1}^2 = V_i^2 + (V_i^p)^2 + (\omega_a r)^2 + 2\omega_a r(V_i^p\cos\theta_1 - V_i\sin\theta_1) \quad (10)$$

where θ_1 is the angle of the blades of the stator. The angle $\alpha + \alpha_1$ can be readily evaluated from Fig. 5.

V_i fluctuates periodically around the mean inflow velocity, $\bar{V}_i$. For the k^{th} blade, this fluctuation consists of the random part, $\alpha_k(t)P_{2_k}(t)$, and the deterministic part, $P_{1_k}(t)$. On the other hand, V_i^p fluctuates randomly around its mean value, which is zero.

By expanding the expression for the force applied to the k^{th} blade, $F_k(t)$, around the point $(\bar{V}_i, 0)$ and by neglecting higher than second-order terms, we obtain the following expression:

$$F_k(t) = F(V_{i_k}(t), V_{i_k}^p(t))$$
$$= \bar{F} + \Delta F_i(V_{i_k}(t) - \bar{V}_i) + \Delta F_i^p V_{i_k}^p$$

where

$$\bar{F} = F(\bar{V}_i, 0) \quad (11)$$

ΔF_i and ΔF_i^p are vectors whose entries are the derivatives of F_k with respect to the components of the velocity vectors V_i and V_i^p, respectively. These derivatives are calculated at the point $(\bar{V}_i, 0)$.

Note that the above quantities are the same for all the blades. Using the expressions for the velocities V_{ik} and V_{ik}^p from Eq. (3), the linearized lift force to the k^{th} blade can be written as follows:

$$F_k(t) = \bar{F} + \Delta F_i(\tilde{P}_{1_k}(t) + \alpha_k(t)P_{2_k}(t)) + \Delta F_i^p\gamma_k(t)P_{2_k}(t) \quad (12)$$

where $\tilde{P}_{1_k}(t)$ is the time varying part of the periodic function $P_{1_k}(t)$, i.e.,

$$\tilde{P}_{1_k}(t) = P_{1_k}(t) - \bar{V}_i \quad (13)$$

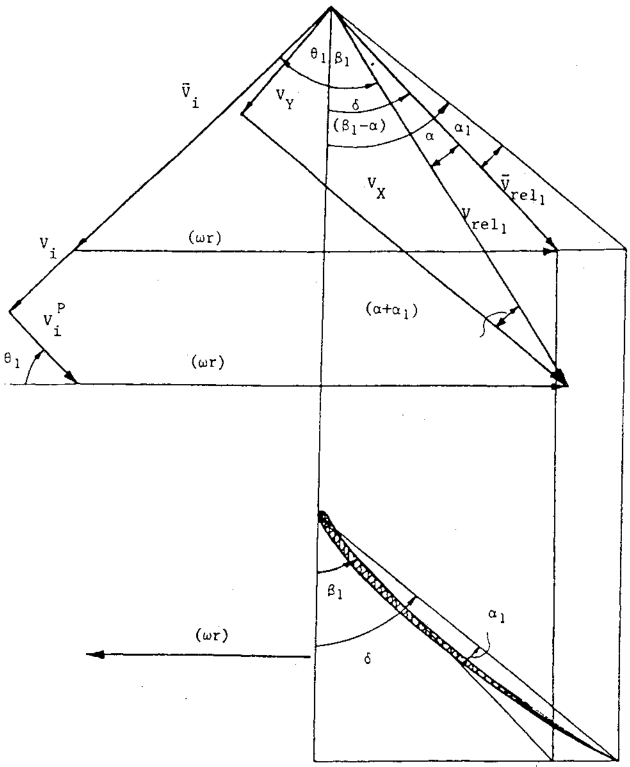

Fig. 5 Inflow velocity diagrams.

The above linearized expressions for the blade force can be used to evaluate the cross-correlation between the forces applied to two blades

$$R_{F_\ell F_k}(t_1, t_2) = (\bar{F})^2 + \Delta F_i^2 \bar{P}_{1_k}(t_1)\bar{P}_{1_\ell}(t_2)$$
$$+ \Delta F_i^2 P_{2_k}(t_1)P_{2_\ell}(t_2)R_{\alpha_k\alpha_\ell}(t_1 - t_2)$$
$$+ (\Delta F_i^P)^2 P_{2_k}(t_1)P_{2_\ell}(t_2)R_{\gamma_k\gamma_\ell}(t_1 - t_2) \tag{14}$$

It is interesting to plot the random part of the cross-correlation of the forces applied to two blades, $R^R_{F_\ell F_k}(t, \tau)$, as a function of the time $t = t_1$ and the elapsed time $\tau = t_1 - t_2$. The decorrelation time and distance for the parallel and perpendicular components of the velocity are assumed to be equal, which implies that C_1 equals C_3 and C_2 equals C_4. Furthermore, we assume that $\sigma_\alpha = \sigma_\gamma$. Then, $R^R_{F_\ell F_k}(t, \tau)$ is given by

$$R^R_{F_\ell F_k}(t, \tau) = (\Delta F_i^2 + (\Delta F_i^P)^2)P_{2_k}(t + \tau)P_{2_\ell}(t)$$
$$\cdot\ \sigma_\alpha^2 e^{-C_1|\tau|} e^{-C_2\omega_a r\left|\tau - \frac{2\pi(k-\ell)}{\omega_a^N}\right|} \tag{15}$$

Figure 6 is a plot of the cross-correlation between the forces applied to the blades 2 and 1, $R^R_{F_2F_1}(t, \tau)$. The two horizontal axes represent times t and τ normalized by the time required for a rotor blade to travel between two stators. In this study, only the first harmonic of the blade force has been taken into consideration. Therefore, the frequency of the blade excitation equals to the frequency of rotation multiplied by the number of stators. It is observed that since the cross-correlation function is cyclostationary, it fluctuates periodically in t. Note that if the blade excitation were stationary, then the cross-correlation would be constant in t. The frequency of fluctuation is equal to the blade excitation frequency multiplied by two. This can be explained by using Eq. (15). Since both $P_{2_k}(t)$ and $P_{2_\ell}(t)$ are harmonic with frequency equal to the frequency of rotation multiplied by the number of blades, their product fluctuates in t with twice that frequency. Note

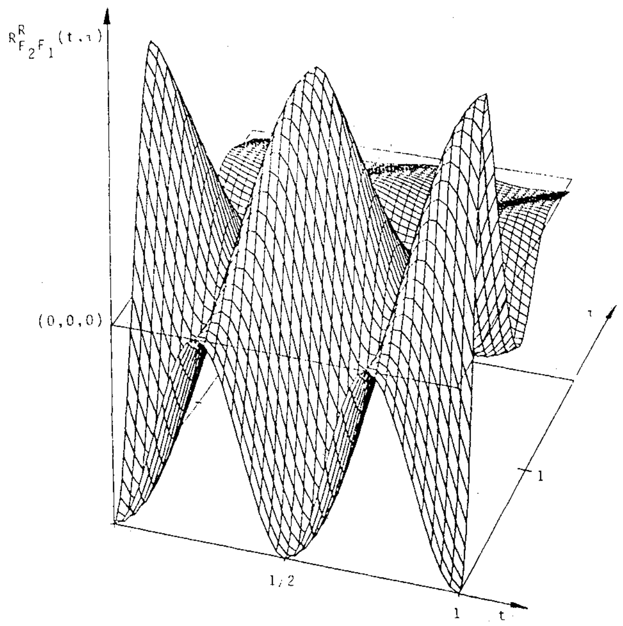

Fig. 6 Cross-correlation, $R^R_{F_2F_1}(t,)$, between blade forces 2 and 1.

that the period of this fluctuation is equal to the time required for a rotor blade to travel between two stators divided by two.

Consider the behavior of the cross-correlation with respect to τ, which is the elapsed time between measuring the forces on blades 2 and 1. Consider that the force on blade 2 is measured when this blade is exactly behind a stator blade. This case corresponds to $t = 0$. When τ equals zero, the force on blade 2 is measured simultaneously with that on blade 1. At this time, the second blade is between two stators (Fig. 7). Therefore, blade 1 experiences a force which fluctuates intensively, while the random component of the force on blade 2 is small. As a result, the two forces are negatively correlated and thus the cross-correlation is negative at t and τ equal to zero. When τ increases to one half while $t = 0$, the force on blade 2 is measured at a time that blade 2 is exactly behind the stator blade. Thus, both the forces on blades 1 and 2 fluctuate intensively and they are positively correlated because they are measured when the two blades pass through the same position. The degree of this positive correlation depends on the length of the decorrelation time of the turbulent flow field. When τ becomes one, the random part of the force on blade 2 becomes small again, since the blade is between two stators. Therefore, the cross-correlation is negative.

Finally, it is observed that the cross-correlation decays with the time difference τ increasing, because the correlation of the velocity field decays exponentially with respect to both the spatial coordinate θ and the time.

Evaluation of the Response: The Input-Output Problem

In this section, we evaluate the response of the blades, which is the tip displacement. The problem consists of determining the statistical properties of the output of a linear system driven by multiple, random inputs. The linear system is the bladed-disk assembly, inputs are the forces, and outputs are the displacements of the blades. The solution of the problem is based on the following assumptions:

1) For the nonresonant case, the displacement of the blades is negligible compared with the displacement under resonant conditions.

2) In case of resonance, the nonresonant harmonics of the excitation can be neglected.

The above assumptions have been made in several studies of the dynamic response of bladed-disk assemblies.[17]

Under these assumptions, the excitation on the blade forces can be considered as Gaussian, amplitude modulated (AM) processes with a fixed phase relationship between the carriers. The carrier frequency equals the product of the frequency of rotation of the rotor by the number of stators. In case of resonance, this carrier frequency is equal to the natural frequency of the system. The modulated amplitudes are Gaussian processes whose autocorrelation and cross-correlation functions have been already established.

The approach, which is used here to determine the output statistics, utilizes the fact that the input processes are wide sense cyclostationary.[8,10,18] In Ref. 10, the problem was solved in its general form by expressing the double Fourier transform of the output autocorrelation function in terms of the double Fourier transform of the input autocorrelation and cross-correlation function. It was shown that the response is also cyclostationary. Furthermore, the frequency of the second moment of the response was found to be equal to two times the frequency of the excitation.

A 3-DOF model of the bladed-disk assembly has been used by Griffin and Hoosac in Ref. 17. A simpler 2-DOF model, which is depicted in Figure 8, is used. Here each blade, and disk segment attached to the blade, are represented by two masses m_1 and m_2, connected by spring K_1. Spring K_6 is used to represent coupling between blades. Finally, the coordinate system for measuring displacements of a blade rotates with it.

Consider the case that the fundamental frequency of the carrier of the random part of the blade excitation, ω_0, coincides with the first natural frequency of the bladed-disk assembly. Then, using assumptions 1) and 2), we can recast the equation for the force to the k^{th} blade as follows:

$$F_k(t) = F_k^D(t) + F_k^R(t) \tag{16}$$

where

$$F_k^D(t) = \bar{F} + \Delta F_i(C_{1_k} e^{j\omega_0 t} + C_{1_k}^* e^{-j\omega_0 t}) \tag{17}$$

is the deterministic part and

$$F_k^R(t) = \Delta F_i \alpha_k(t)(C_{2_k} e^{j\omega_0 t} + C_{2_k}^* e^{-j\omega_0 t}) + \Delta F_i^P \gamma_k(t)(C_{2_k} e^{j\omega_0 t} + C_{2_k}^* e^{-j\omega_0 t}) \tag{18}$$

is the random part. C_{1_k} and C_{2_k} are the Fourier coefficients of periodic functions $P_{2_k}(t)$ and $P_{2_k}(t)$, respectively.

The response of a blade is the summation of a deterministic and a random part. The mean value of the random part is zero. Using the results of Ref. 10, we evaluate the blade response. The following expression is found for the autocorrelation function of the random part:

$$R_r^R(t+\tau, t) = (\Delta F_i)^2 2Re\{A(\tau)e^{j\omega_0\tau} + B(\tau)e^{j\omega_0\tau}e^{j2\omega_0 t}\} + (\Delta F_i^P)^2 2Re\{X(\tau)e^{j\omega_0\tau} + Y(\tau)e^{j\omega_0\tau}e^{j2\omega_0 t}\}$$

where

$$A(\tau) = \frac{1}{2\pi}\sum_{\ell=1}^{N}\sum_{k=1}^{N}\int_{-\infty}^{+\infty} \cdot C_{2_\ell}C_{2_k}^* S_{\alpha_\ell\alpha_k}(\omega_3)H_\ell(\omega_3+\omega_0)H_k^*(\omega_3+\omega_0)e^{j\omega_3\tau}d\omega_3$$

and

$$B(\tau) = \frac{1}{2\pi}\sum_{\ell=1}^{N}\sum_{k=1}^{N}\int_{-\infty}^{+\infty} \cdot C_{2_\ell}C_{2_k}^* S_{\alpha_l\alpha_k}(\omega_3)H_l(\omega_3+\omega_0) \cdot H_k^*(\omega_3-\omega_0)e^{j\omega_3\tau}d\omega_3 \tag{19}$$

$H_i(\omega)$ is the transfer function corresponding to the force applied to the i^{th} blade. The expression for $X(\tau)$ and $Y(\tau)$ can be obtained from $A(\tau)$ and $B(\tau)$, respectively, by replacing α by γ. $S_{\alpha_l\alpha_k}(\omega)$ is obtained by taking the Fourier transform of the autocorrelation function in Eq. (5).

The spectrum corresponding to $\gamma(t)$ is of the same form, except that parameters C_1 and C_2 are replaced by C_3 and C_4, respectively.

The displacement of the blades is a Gaussian process, and as such, it can be completely described by its mean and au-

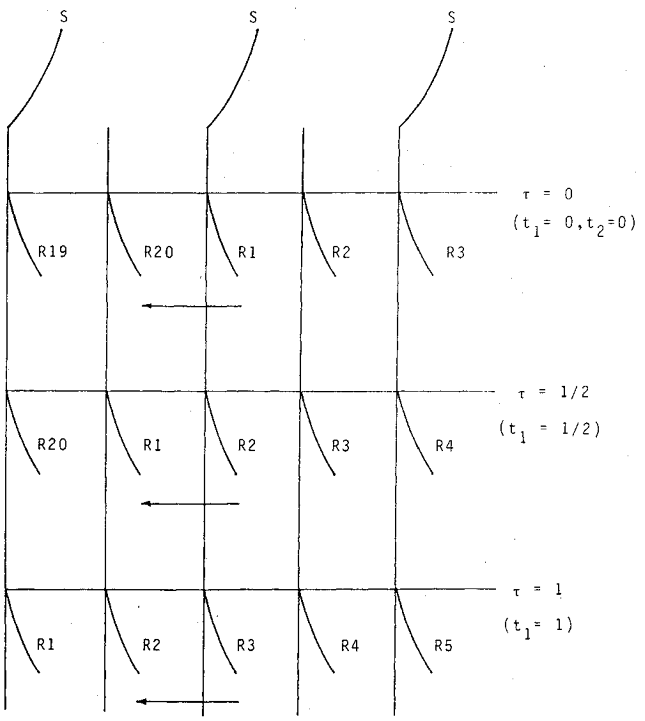

Fig. 7 Position of rotor blades relative to stator blades.

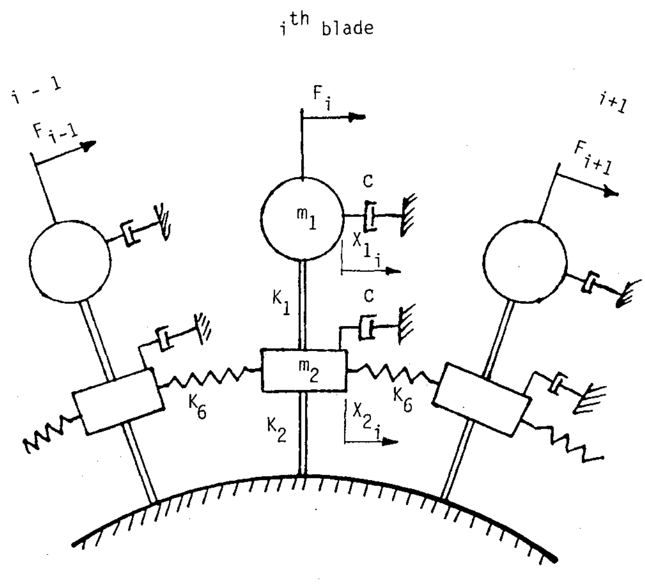

Fig. 8 Dynamic model of a bladed-disk assembly.

tocorrelation function. Therefore, the above results are sufficient for studying the response of bladed-disk assemblies.

The rms of the deterministic part is

$$\text{rms}^D = (\Delta F_1)2Re \sum_{\ell=1}^{N} \sum_{k=1}^{N} \{C_{1\ell} C_{1k} H_\ell(\omega_0) H_k(\omega_0) e^{j2\omega_0 t} + C_{1\ell} C_{1k}^* H_\ell(\omega_0) H_k^*(\omega_0)\} \tag{20}$$

It is observed that, as in the case of the random part, the rms fluctuates with frequency equal to two times the frequency of excitation.

Examples

In this section, we present examples of evaluating the response of bladed-disk assemblies due to random turbulent excitation.

Let us consider a low-speed, single-stage, air-driven turbine. The rotor consists of 20 blades and there are 10 stator blades. The mean inflow velocity ranges between 51.29 m/s to 57.90 m/s. The first natural frequency of the rotor is 2709 rad/s. More data for this turbine can be found in Table 1.

The driving frequency has been assumed to be in the range from 2400.0 rad/s to 2709.1 rad/s, which corresponds to rotational velocities from 2291.8 rpm up to 2587.0 rpm. When the turbine operates at 2400.0 rad/s, there is no resonance but as the frequency increases approaching 2709.1 rad/s, we approach resonance. It is unlikely that a turbine will continuously operate at a speed corresponding to the natural frequency of the assembly because this speed is too low. However, resonance may be encountered during transient operation.

We assume that the flow in front of the stator is turbulent and that the scale of turbulence is comparable to the size of the blades. The selection of values of the velocities and blade angles has been based on the data presented in Refs. 2, 19, 20. The values for velocities and blade angles, which are given in Table 1, have been used to evaluate the linearization constants $\bar{F}$, ΔF_1 and ΔF_i^p in Eq. (14).

Equation (19) has been used to evaluate the autocorrelation function of the response due to the random part of the excitation. The rms of the random and deterministic parts of the response are compared in order to assess the importance of considering the former in the analysis. The constant part of the response has been dropped since we are not interested in the steady deflection of the blades.

We have also calculated the blade response by using a conventional approach which treats the excitation as stationary. The rms of the stationary process, which is used to model the excitation, is equal to the temporal average of the rms of the cyclostationary excitation over one period of its fluctuation. The autocorrelation and cross-correlation of the blade forces are

$$R^R_{F_\ell F_k}(\tau) = \frac{1}{2}(\Delta F_i + (\Delta F_i^p)^2) \cdot P_2(\tau - \theta_k + \theta_\ell)\sigma_\alpha^2 e^{-C_1|\tau| - C_2\omega_a r\left|\frac{\tau - 2\pi(k-\ell)}{\omega_a N}\right|} \tag{21}$$

This relation was derived from Eq. (15), which corresponds to the cyclostationary case, by finding the average value of the autocorrelation and the cross-correlation over time.

The rms of the deterministic part is plotted in Fig. 9. (All values given in Figs. 9–11 have been normalized with respect to the maximum value of the rms of the deterministic part.) This is a harmonic function with frequency equal to two times the frequency of excitation. Since the frequency of excitation is ten times the frequency of rotation, the frequency of the rms of the instantaneous velocity is equal to twenty times the rotational frequency. Therefore, one cycle of oscillation of the rms of the deterministic part is completed over an angle of rotation that is 18 deg.

Table 1 Turbine data

N	=	20 rotor blades
S	=	10 stator blades
r	=	0.2351 m
ω_a	=	240.0–270.9 rad/s
ω_0	=	2400.0–2709.1 rad/s
$\bar{V}_i$	=	51.29 m/s–57.90 m/s
$\overline{V_i^p}$	=	0 m/s
θ_1	=	60°
β_1	=	18.43 deg
β	=	5 deg
α_1	=	0 deg
$\pi\rho\ell$	=	1 kg/m^2

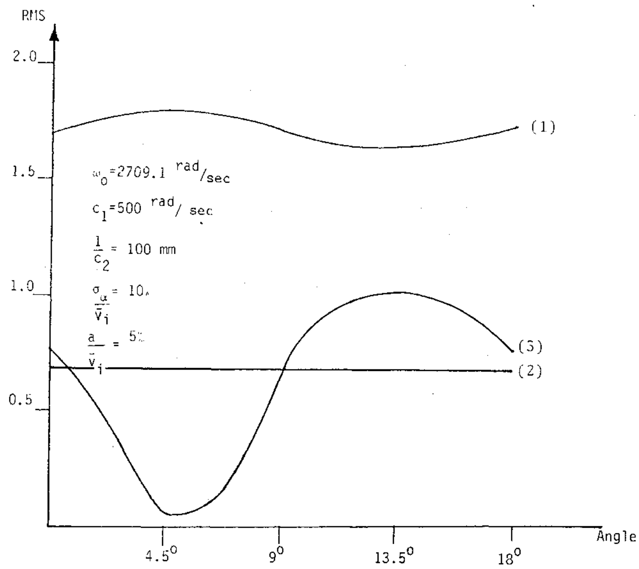

Fig. 9 RMS found using 1) cyclostationary, 2) stationary, and 3) deterministic approaches.

The rms of the random part of the response, which is calculated by treating the excitation as cyclostationary, was calculated by taking the square root of its autocorrelation function evaluated at τ equal to zero. This last quantity can be evaluated using Eq. (19) for $\tau = 0$. In this example we have selected the same spectrum and parameter values for the second moments of the random processes $\alpha(t)$ and $\gamma(t)$, which are used to account for the randomness in the parallel inflow and perpendicular velocities.

The rms of the random part of a blade response is plotted in Fig. 9. This quantity changes periodically with the angle of rotation of the blade, and therefore with time, since the response is cyclostationary. As in the case of the deterministic part, one cycle of fluctuation of the random part corresponds to an angle of 18 deg. Coefficient C_2 was assumed to be 1/100 mm^{-1}. This corresponds to a decorrelation distance for the velocity of 100 mm. The value of C_1 is 500 rad/s. It is observed that the highest value of the rms of the random part, which corresponds to $C_1 = 500$ rad/s, is approximately 176% of that of the deterministic component. Moreover, the rms of the response, which was calculated by the simplified stationary approach, is significantly smaller than that of the random part, which was calculated by the cyclostationary approach.

The maximum value of the response rms versus the frequency of turbulence is plotted in Fig. 10. The ratio of the rms of the random and the deterministic part decreases from almost 200% at 271 rad/s to 55% at 10,000 rad/s. This trend can be explained as follows. When C_1 increases, the spectrum of the turbulent excitation flattens out and the energy con-

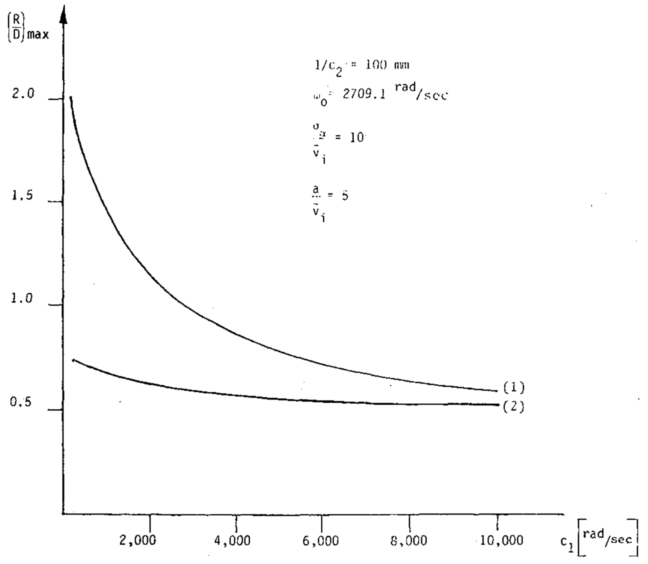

Fig. 10 Ratio of the rms of the 1) cyclostationary and deterministic approaches, and 2) stationary and deterministic approaches, as a function of frequency of turbulence.

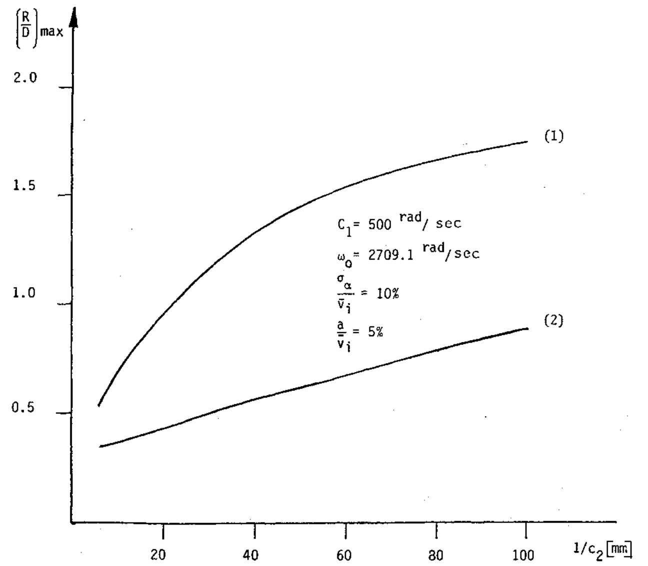

Fig. 11 Ratio of the rms of the 1) cyclostationary and deterministic approaches and 2) stationary and deterministic approaches as a function of size of eddies.

tained in the portion of the spectrum corresponding to low frequencies in the range of the natural frequency of the bladed-disk assembly is reduced. Since the response is due to the excitation in this low range of frequencies, its intensity becomes lower for higher values of C_1. The dependence of the intensity of the random part of the response on the scale of turbulence is studied in Fig. 11. We observe that the random part of the response increases with the size of turbulence increasing. Finally, the response which corresponds to the cyclostationary model is larger than that corresponding to the stationary approach for all cases in Figs. 9–11.

The above results indicate that the rms is due to the random part of the excitation, and may be significantly larger than that of the deterministic part. Furthermore, an approach which treats the blade excitation as stationary is likely to underestimate the response. Besides underestimating the rms of the blade response, a simplified stationary approach underestimates the risk of structural failure. The reason is that the probabilities of failure due to first excursion of a maximum allowable threshold by the stress and due to fatigue are very sensitive to the maximum rms of the response.[10]

The above observations lead to the conclusion that the random part of the aerodynamic excitation might be important and that it should be modeled as a cyclostationary process.

Concluding Remarks

An analysis of random vibrations of bladed-disk assemblies due to turbulent excitation is presented. The blade forces were modeled as nonstationary processes with statistics varying periodically in time.

The cyclostationary model, which is presented here, represents the physics of the problem more accurately than a model which treats the excitation as stationary. It was demonstrated that the response to turbulent excitation might be considerable compared to that due to deterministic excitation. Moreover, a model which treats the excitation as stationary is likely to underestimate the response of the blades and the risk of structural failure. Consequently, for cases that the turbulent aerodynamic excitation must be considered, a cyclostationary model is more appropriate than a stationary one.

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